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Golden rule decay versus Lyapunov decay of the quantum Loschmidt echo

Ph. Jacquod,[1] P.G. Silvestrov,[1,2] and C.W.J. Beenakker[1]
[1]*Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands*
[2]*Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia*

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The overlap of two wave packets evolving in time with slightly different Hamiltonians decays exponentially $\propto e^{-\gamma t}$, for perturbation strengths U greater than the level spacing Δ. We present numerical evidence for a dynamical system that the decay rate γ is given by the *smallest* of the Lyapunov exponent λ of the classical chaotic dynamics and the level broadening U^2/Δ that follows from the golden rule of quantum mechanics. This implies the range of validity $U > \sqrt{\lambda\Delta}$ for the perturbation-strength independent decay rate discovered by Jalabert and Pastawski [Phys. Rev. Lett. **86**, 2490 (2001)].

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The search for classical Lyapunov exponents in quantum mechanics is a celebrated problem in quantum chaos [1]. Motivated by NMR experiments on spin echoes [2], Jalabert and Pastawski [3] have given analytical evidence, supported by computer simulations [4], that the Lyapunov exponent governs the time dependence of the fidelity

$$M(t) = |\langle \psi | \exp(iHt) \exp(-iH_0 t) | \psi \rangle|^2, \tag{1}$$

with which a wave packet ψ can be reconstructed by inverting the dynamics with a perturbed Hamiltonian $H = H_0 + H_1$. They have called this the problem of the ''quantum Loschmidt echo.'' The fidelity $M(t)$ can equivalently be interpreted as the decaying overlap of two wave functions that start out identically and evolve under the action of two slightly different Hamiltonians, a problem first studied in perturbation theory by Peres [5].

Perturbation theory breaks down once a typical matrix element U of H_1 connecting different eigenstates of H_0 becomes greater than the level spacing Δ. Then the eigenstates of H, decomposed into the eigenstates of H_0, contain a large number of non-negligible components. The distribution $\rho(E)$ (local spectral density) of these components over energy has a Lorentzian form

$$\rho(E) = \frac{\Gamma}{2\pi(E^2 + \Gamma^2/4)}, \tag{2}$$

with a spreading width $\Gamma \simeq U^2/\Delta$ given by the golden rule [6,7]. A simple calculation in a random-matrix model gives an average decay $\overline{M} \propto \exp(-\Gamma t)$ governed by the same golden rule width. This should be contrasted with the exponential decay $\overline{M} \propto \exp(-\lambda t)$ obtained by Jalabert and Pastawski [3], which is governed by the Lyapunov exponent λ of the classical chaotic dynamics.

Since the random-matrix model has by construction an infinite Lyapunov exponent, one way to unify both results would be to have an exponential decay with a rate set by the smallest of Γ and λ. We will in what follows present numerical evidence for this scenario, using a dynamical system in which we can vary the relative magnitude of Γ and λ. There exists a third energy scale, the inverse of the Ehrenfest time τ_E, that is smaller than the Lyapunov exponent by a factor logarithmic in the system's effective Planck constant. In our numerics we do not have enough orders of magnitude between $1/\tau_E$ and λ to distinguish between the two, so that our findings remain somewhat inconclusive in this respect.

Because Γ cannot become bigger than the band width B of H_0 (we are interested in the regime $H_1 < H_0$), a consequence of a decay $\overline{M} \propto \exp[-t \min(\lambda,\Gamma)]$ is that the regime of Lyapunov decay can only be reached with increasing U if λ is considerably less than B. That would exclude typical fully chaotic systems, in which λ and B are comparable, and set limits of observability of the Lyapunov decay.

The crossover from the golden rule regime to a regime with a perturbation-strength independent decay, obtained here for the Loschmidt echo, should be distinguished from the corresponding crossover in the local spectral density $\rho(E)$, obtained by Cohen and Heller [8]. The Fourier transform of $M(t)$ would be equal to $\rho(E)$ if ψ would be an eigenstate of H_0 rather than a wave packet. The choice of a wave packet instead of an eigenstate does not matter in the golden rule regime, but is essential for a decay rate given by the Lyapunov exponent.

The dynamical model that we have studied is the kicked top [9], with Hamiltonian

$$H_0 = (\pi/2\tau) S_y + (K/2S) S_z^2 \sum_n \delta(t - n\tau). \tag{3}$$

It describes a vector spin (magnitude S) that undergoes a free precession around the y axis perturbed periodically (period τ) by sudden rotations around the z axis over an angle proportional to S_z. The time evolution of a state after n periods is given by the nth power of the Floquet operator

$$F_0 = \exp[-i(K/2S) S_z^2] \exp[-i(\pi/2) S_y]. \tag{4}$$

Depending on the kicking strength K, the classical dynamics is regular, partially chaotic, or fully chaotic. The dependence of the Lyapunov exponent λ on K is plotted in the inset to Fig. 1 (cf. Ref. [10]). The error bars reflect the spread in λ in different regions of phase space, in particular the presence of islands of stability. For $K \gtrsim 9$ the error bars vanish because the system has become fully chaotic. For the reversed time

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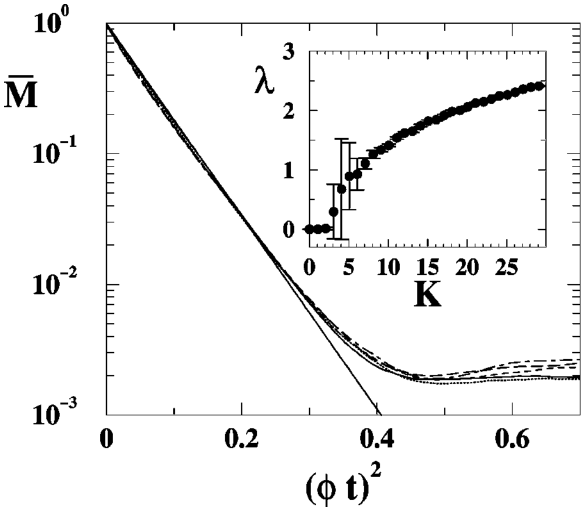

FIG. 1. Decay of the average fidelity $\bar{M}$ for the quantum kicked top with $K=13.1$ and $S=500$, as a function of the squared rescaled time $(\phi t)^2$. The perturbation strengths range between $\phi=10^{-7}$ and 10^{-6}. The straight line corresponds to the Gaussian decay (6) valid in the perturbative regime. Inset: Numerically computed Lyapunov exponent for the classical kicked top as a function of the kicking strength K. Dots correspond to averages taken over 10^4 initial conditions (see Ref. [10]). The error bars reflect different results obtained with different initial conditions. The vanishing of error bars indicates the disappearance of islands of regular dynamics.

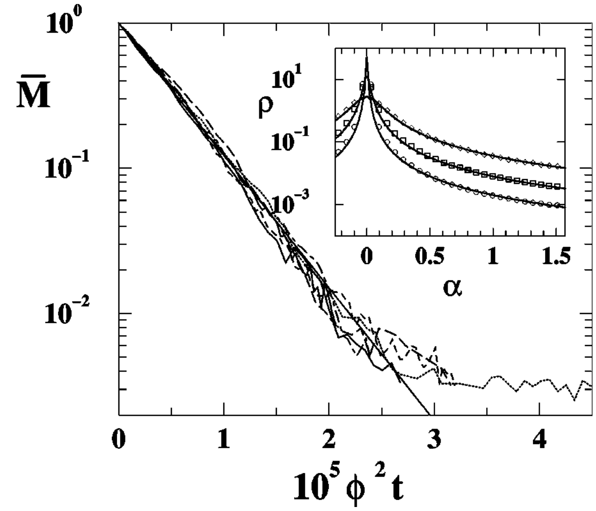

FIG. 2. Decay of $\bar{M}$ in the golden rule regime for kicking strengths $K=13.1$, 17.1, and 21.1 as a function of the rescaled time $\phi^2 t$. Perturbation strengths range from $\phi=10^{-4}$ to 10^{-3}. Inset: Local spectral density of states for $K=13.1$ and perturbation strengths $\phi=2.5\times10^{-4},5\times10^{-4},10^{-3}$. The solid curves are Lorentzian fits, from which the decay rate $\Gamma\approx0.84\phi^2S^2$ is extracted. The solid line in the main plot gives the decay $\bar{M}\propto\exp(-\Gamma t)$ with this value of Γ.

evolution we introduce as a perturbation a periodic rotation of constant angle around the x axis, slightly delayed with respect to the kicks H_0,

$$H_1=\phi S_x\sum_n \delta(t-n\tau-\epsilon). \tag{5}$$

The corresponding Floquet operator is $F=\exp(-i\phi S_x)F_0$. We have set $\hbar=1$ and in what follows we will also set $\tau=1$ for ease of notation.

Both H and H_0 conserve the spin magnitude. We choose the initial wave packets as coherent states of the spin SU(2) group [11], i.e., states that minimize the Heisenberg uncertainty in phase space (in our case on a sphere of fixed radius) at the effective Planck constant $h_{\rm eff}\sim S^{-1}$. The corresponding Ehrenfest time is $\tau_{\rm E}=\lambda^{-1}\ln S$ [12]. We take $S=500$ and average $M(t=n)=|\langle\psi|(F^\dagger)^nF_0^n|\psi\rangle|^2$ over 100 initial coherent states ψ.

We first show results in the fully chaotic regime $K>9$, where we choose the initial states randomly over the entire phase space. The local spectral density $\rho(\alpha)$ of the eigenstates of F (in the basis of the eigenstates of F_0 with eigenphases α) is plotted for three different ϕ's in the inset to Fig. 2. The curves can be fitted by Lorentzians from which we extract the spreading width Γ. (It is given up to numerical coefficients by $\Gamma\simeq U^2/\Delta$, $U\simeq\phi\sqrt{S}$, $\Delta\simeq1/S$). The golden rule regime $\Gamma\gtrsim\Delta$ is entered at $\phi_c\approx1.7\times10^{-4}$. For $\phi\ll\phi_c$ we are in the perturbative regime, where eigenstates of F do not appreciably differ from those of F_0 and eigenphase differences can be calculated in first-order perturbation theory. We then expect the Gaussian decay

$$\bar{M}\propto\exp(-U^2t^2)\Rightarrow\ln\bar{M}\propto(\phi t)^2. \tag{6}$$

This decay is evident in Fig. 1, which shows $\bar{M}$ as a function of $(\phi t)^2$ on a semilogarithmic scale for $\phi\leqslant10^{-6}$. The decay (6) stops when $\bar{M}$ approaches $M_\infty=1/2S$, being the inverse of the dimension of the Hilbert space. This saturation reflects the finiteness of the system and eventually prevails at long times independently of the strength of the perturbation.

For $\phi>\phi_c$ one enters the golden rule regime, where the Lorentzian spreading of eigenstates of F over those of F_0 results in the exponential decay

$$\bar{M}\propto\exp(-U^2t/\Delta)\Rightarrow\ln\bar{M}\propto\phi^2t. \tag{7}$$

The data presented in Fig. 2 clearly confirm the validity of the scaling (7). There is no dependence of $\bar{M}$ on K in this regime of moderate (but nonperturbative) values of ϕ, i.e., no dependence on the Lyapunov exponent (λ varies by a factor of 1.4 for the different values of K in Fig. 2).

We cannot satisfy $\lambda<\Gamma$ in the fully chaotic regime, for the reason mentioned in the Introduction: The band width B (which is an upper limit for Γ) is $B=\pi/2$ (in units of $1/\tau$), while $\lambda\gtrsim1$ for fully developed chaos in the kicked top (see the inset to Fig. 1). For this reason, when the perturbation strength ϕ is further increased, the golden rule decay rate saturates at the bandwidth — before reaching the Lyapunov exponent. This is shown in Fig. 3. There is no trace of a Lyapunov decay in this fully chaotic regime.

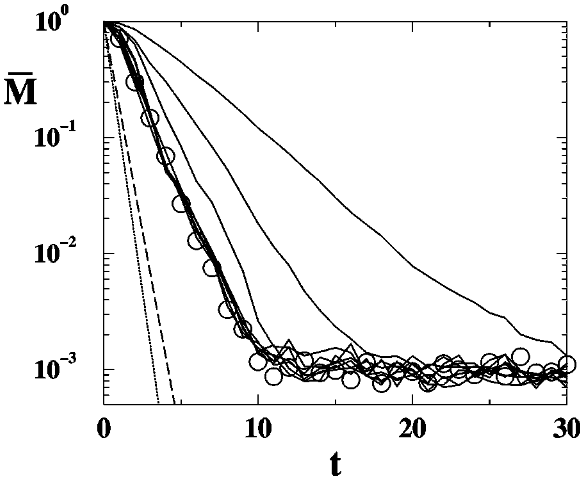

FIG. 3. Decay of $\bar{M}$ in the golden rule regime without rescaling of time, for $K=13.1$, $\phi=j\times10^{-3}$, $(j=1,1.5,2,\ldots 5)$ (solid curves) and $K=21.1$, $\phi=3\times10^{-3}$ (circles). Dashed and dotted lines show exponential decays with Lyapunov exponents $\lambda=1.65$ and 2.12, corresponding to $K=13.1$ and 21.1, respectively. The decay slope saturates at $\phi\approx2.5\times10^{-3}$, when Γ reaches the bandwidth.

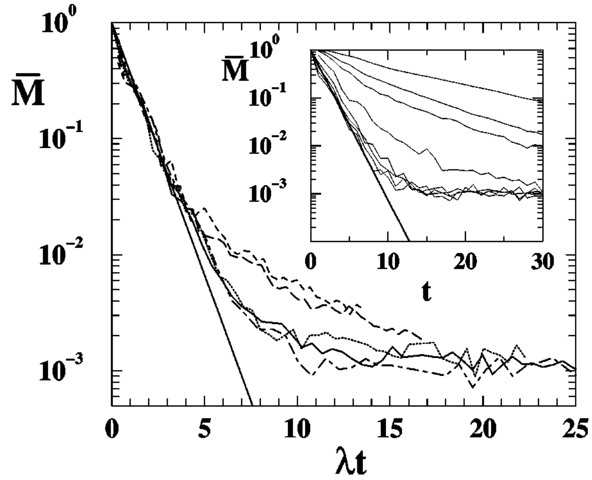

FIG. 4. Decay of $\bar{M}$ in the Lyapunov regime, for $\phi=2.1\times10^{-3}$, $K=2.7,3.3,3.6,3.9,4.2$. The time is rescaled with the Lyapunov exponent λ, ranging from 0.22–0.72. The straight solid line indicates the decay $\bar{M}\propto\exp(-\lambda t)$. Inset: $\bar{M}$ for $K=4.2$ and different $\phi=j\times10^{-4}$, $j=1,2,3,4,5,9,17,25$. The decay slope saturates at the value $\phi\approx1.7\times10^{-3}$ for which $\Gamma\approx\lambda$, even though Γ keeps on increasing. This demonstrates the decay law $\bar{M}\propto\exp(-\gamma t)$ with $\gamma=\min(\Gamma,\lambda)$.

We therefore reduce K to values in the range $2.7\leqslant K\leqslant4.2$, which allows us to vary the Lyapunov exponent over a wider range between 0.22 and 0.72. In this range the classical phase space is mixed and we have coexisting regular and chaotic trajectories. We choose the initial coherent states in the chaotic region (identified numerically through the participation ratio). Because the chaotic region still occupies more than 80% of the phase space for the smallest value of K considered, nonuniversal effects (e.g., nonzero overlap of our initial wavepackets with regular eigenfunctions of F_0 or F) should be negligible. We expect a crossover from the golden rule decay (7) to the Lyapunov decay [3]

$$\bar{M}\simeq\exp(-\lambda t)\Rightarrow\ln\bar{M}\propto\lambda t, \tag{8}$$

once Γ exceeds λ. This expectation is borne out by our numerical simulations, see Fig. 4.

In conclusion, we have presented numerical evidence for the existence of three distinct regimes of exponential decay of the Loschmidt echo: the perturbative regime (6), the golden rule regime (7), and the Lyapunov regime (8). The perturbation strength independent decay in the Lyapunov regime is reached in our simulation if $\lambda<\Gamma$, which prevents its occurrence for fully developed chaos in the model considered here. Our numerics are limited by a relatively small window between λ and $1/\tau_E$ (a factor $\ln S\approx6$). It remains to be seen if the Lyapunov decay can be observed under conditions of fully developed chaos and $\Gamma<\lambda$ by increasing S so that $1/\tau_E$ becomes larger than Γ. It is noteworthy that for a Lyapunov decay $\bar{M}\propto\exp(-\lambda t)$, the saturated fidelity $M_\infty=1/2S$ is reached at the Ehrenfest time τ_E (as can also be seen in Fig. 4), so that a Lyapunov decay for $t\lesssim\tau_E$ rules out golden rule decay for later times. Similar investigations in strongly chaotic systems with small Lyapunov exponents (like the Bunimovich stadium with short straight segments) are highly desirable.

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